

Full-time dynamics of modulational instability in spinor Bose–Einstein condensates

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We describe the full-time dynamics of modulational instability in $F = 1$ spinor Bose–Einstein condensates for the case of the integrable three-component model associated with the matrix nonlinear Schrödinger equation. We obtain an exact homoclinic solution of this model by employing the dressing method which we generalize to the case of the higher-rank projectors. This homoclinic solution describes the development of modulational instability beyond the linear regime, and we show that the modulational instability demonstrates the reversal property when the growth of the modulated amplitude is changed by its exponential decay.

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I. INTRODUCTION

Spinor Bose–Einstein condensate (BEC) with an optical confinement represents a unique macroscopic system with the spin degrees of freedom [1, 2]. The interplay between the mean-field effective nonlinearities of three-component matter waves and their spin properties produce many interesting phenomena such as the spin mixing [2], as well as the formation of spin domains [3, 4] and spin textures [5, 6]. Various properties of the spinor BEC have been analyzed theoretically [7, 8, 9]. The ground state of the spinor BEC with the hyperfine spin $F = 1$ can be either ferromagnetic (maximum spin projection) or polar (zero spin projection). It was shown in Ref. [10] within the linear stability analysis of the spinor condensate model that the ferromagnetic phase of the condensate can experience instability for large enough densities of atoms, while the polar phase remains always modulationally stable.

Wadati and co-authors [11] demonstrated that the three-component nonlinear equations describing the evolution of the $F = 1$ BEC can be reduced, under special constraints imposed on the condensate parameters, to the completely integrable matrix nonlinear Schrödinger (NLS) equation [12]. Both bright and dark three-component BEC solitons have been found in the framework of this model [13, 14, 15, 16, 17].

As regards the linear stability analysis presented in Ref. [10], only an initial (linear) stage of the perturbation development can be explored by this method which predicts the exponential growth of the modulation frequency sidebands for some conditions, i.e., it describes the conditions of modulational instability (MI). A physical mechanism behind the MI is the parametric coupling between the spin degrees of freedom which leads to a pop-

ulation transfer between the spin components. To study the long-time evolution of instabilities, numerical methods are used as a rule. For the scalar NLS equation, the problem of the long-time evolution of the MI was studied by the truncation of the original model to a finite number of modes (as usually, the three-mode approximation) [18]. More complete analysis of the long-time MI dynamics [19, 20] is based on a linear constraint imposed on the real and imaginary parts of solutions of the scalar NLS equation, and it allows one to find a class of three-parameter solutions sharing this property. Among the solutions found in such a way, a special solution describes the development of MI beyond the linear regime, and it is identified as a homoclinic orbit separating two qualitatively different types of periodic solutions. A similar result was obtained by means of the Darboux transformation with the plane wave as a ‘seed’ solution [21]. Following terminology of Ref. [22], the full-time dynamics represents a *nonlocal* view of the MI development over a long time interval.

A homoclinic orbit is a trajectory of a dynamical system that tends to the same manifold (fixed point, periodic orbit, etc.) as time tends to $\pm\infty$. The existence of homoclinic solutions serves as an indicator of chaotic regimes in a perturbed deterministic system. For nonlinear wave systems described by partial differential equations, the complete understanding of the homoclinic structures in the infinite-dimensional phase space is far from being available at present. On the other hand, the unique features of the integrable nonlinear wave equations admit essentially more deep insight into this problem. Extended reviews of analytical and numerical methods for obtaining homoclinic orbits for the scalar NLS and sine-Gordon equations are given in Refs. [23, 24].

The aim of our paper is twofold. First, we derive a homoclinic solution of the matrix NLS equation. Second, using these analytic results, we present the exact solution of the problem of the long-time evolution of the modulationally unstable $F = 1$ BEC in the case when it

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is described by the integrable model.

To find homoclinic solutions, we do not impose *ad hoc* constraints on the form of solutions. Instead, we use a kind of dressing procedure, well known in the soliton theory [25], which was proposed recently as a systematic tool to generate exact homoclinic solutions of integrable nonlinear equations with periodic boundaries [26]. A dressing factor being the main technical ingredient in this approach contains a projector which determines the coordinate dependence of the homoclinic solution. It should be pointed out that for all known homoclinic solutions obtained up to now for various nonlinear equations (see, e.g., Refs. [26, 27]), this projector has rank 1. A crucial feature of the matrix NLS equation consists in the fact that the corresponding dressing factor incorporates the rank 2 projector. In terms of the soliton theory it corresponds to *multiple* zeros of the scattering matrix coefficients (or multiple zeros of the associated Riemann–Hilbert problem). Notice that the case of multiple zeros cannot be treated as a coalescence of simple zeros [28]. Accordingly, we modify the definition of the dressing factor for the case of the matrix NLS equation and obtain the first example of the *matrix* homoclinic orbit and, as a result, the complete description of the MI evolution in the integrable spinor BEC model.

The paper is organized as follows. In Sec. II we describe the integrable $F = 1$ BEC model. The method for obtaining homoclinic solutions for integrable nonlinear equations valid for higher rank projectors is outlined in Sec. III. Section IV is devoted to the explicit derivation of the homoclinic solution for the matrix NLS equation and presents the main results of our paper. The homoclinic solution describes the temporal evolution of *linearly unstable* modes. We show that the MI has a reversal property – the initial-wave profile is recovered after a sufficiently long time. Hence, the term ‘side-band instability’ refers in fact to only the linear stage of the instability development. Section V concludes the paper.

II. MODEL

We consider an effective one-dimensional BEC trapped in a pencil-shaped region elongated in the x direction and tightly confined in the transversal directions. The assembly of atoms in the hyperfine spin $F = 1$ state is described by a vector order parameter $\vec{\Phi}(x, t) = (\Phi_+(x, t), \Phi_0(x, t), \Phi_-(x, t))^T$, where its components correspond to three values of the spin projection $m_F = 1, 0, -1$. The functions Φ_\pm and Φ_0 obey a system of coupled Gross–Pitaevskii equations [13, 29]

$$\begin{aligned} i\hbar\partial_t\Phi_\pm &= -\frac{\hbar^2}{2m}\partial_x^2\Phi_\pm + (c_0 + c_2)(|\Phi_\pm|^2 + |\Phi_0|^2)\Phi_\pm \\ &\quad + (c_0 - c_2)|\Phi_\mp|^2\Phi_\pm + c_2\Phi_\mp^*\Phi_0, \\ i\hbar\partial_t\Phi_0 &= -\frac{\hbar^2}{2m}\partial_x^2\Phi_0 + (c_0 + c_2)(|\Phi_+|^2 + |\Phi_-|^2)\Phi_0 \\ &\quad + c_0|\Phi_0|^2\Phi_0 + 2c_2\Phi_+\Phi_-\Phi_0^*, \end{aligned} \quad (2.1)$$

where the constant parameters $c_0 = (g_0 + 2g_2)/3$ and $c_2 = (g_2 - g_0)/3$ control the spin-independent and spin-dependent interaction, respectively. The coupling constant g_f ($f = 0, 2$) is given in terms of the s -wave scattering length a_f in the channel with the total hyperfine spin f ,

$$g_f = \frac{4\hbar^2 a_f}{ma_\perp^2} \left(1 - C \frac{a_f}{a_\perp}\right)^{-1}.$$

Here a_\perp is the size of the transverse ground state, m is the atom mass, and $C = -\zeta(1/2) \approx 1.46$.

It was noted in [11] that Eqs. (2.1) are reduced to an integrable system under the constraint

$$c_0 = c_2 \equiv -c < 0. \quad (2.2)$$

The negative c_2 means that we consider the ferromagnetic ground state of the spinor BEC with attractive interactions. The condition (2.2), being written in terms of g_f as $2g_0 = -g_2 > 0$, imposes a constraint on the scattering lengths: $a_\perp = 3Ca_0a_2/(2a_0 + a_2)$. Redefining the function $\vec{\Phi}$ as $\vec{\Phi} \rightarrow (\phi_+, \sqrt{2}\phi_0, \phi_-)^T$, normalizing the coordinates as $t \rightarrow (c/\hbar)t$ and $x \rightarrow (\sqrt{2m\hbar}/\hbar)x$, and accounting for the constraint (2.2), we obtain a reduced system of equations in a dimensionless form:

$$i\partial_t\phi_\pm + \partial_x^2\phi_\pm + 2(|\phi_\pm|^2 + 2|\phi_0|^2)\phi_\pm + 2\phi_\mp^*\phi_0^2 = 0, \quad (2.3)$$

$$i\partial_t\phi_0 + \partial_x^2\phi_0 + 2(|\phi_+|^2 + |\phi_0|^2 + |\phi_-|^2)\phi_0 + 2\phi_+\phi_0^*\phi_- = 0.$$

After arranging the components ϕ_\pm and ϕ_0 into a 2×2 matrix $Q = \begin{pmatrix} \phi_+ & \phi_0 \\ \phi_0 & \phi_- \end{pmatrix}$, we can easily see that Eqs. (2.3) take the form of the integrable matrix NLS equation

$$i\partial_t Q + \partial_x^2 Q + 2QQ^\dagger Q = 0. \quad (2.4)$$

The matrix NLS equation (2.4) possesses the Lax representation with the 4×4 matrices U and V of the form [12]

$$U = ik\Lambda + \hat{Q}, \quad \Lambda = \text{diag}(-1, -1, 1, 1), \quad \hat{Q} = \begin{pmatrix} 0 & Q \\ -Q^\dagger & 0 \end{pmatrix}, \quad (2.5)$$

$$V = 2ik^2\Lambda + 2k\hat{Q} + i \begin{pmatrix} QQ^\dagger & Q_x \\ Q_x^\dagger & -Q^\dagger Q \end{pmatrix}, \quad (2.6)$$

k is a spectral parameter.

III. METHOD

We are interested in periodic solutions of Eqs. (2.3) (or (2.4)) with a spatial period L , $Q(x + L, t) = Q(x, t)$. Hence, the Floquet theory should be applied to analyze the spectral problem

$$M_x = UM. \quad (3.1)$$

The fundamental solution $M(x, k)$ of Eq. (3.1) is fixed by the condition $M(0, k) = I$, I is the unit 4×4 matrix. Then we define a transfer matrix $T(k)$ as the fundamental solution in the point $x = L$, $T(k) = M(L, k)$. Diagonalization of the transfer matrix determines a matrix R ,

$$R^{-1}T(k)R = \text{diag}(e^{im_1L}, \dots, e^{im_4L}) \equiv \Delta(L, k),$$

and produces the Floquet multipliers $\exp(im_jL)$ with the Floquet exponents m_j , $j = 1, \dots, 4$. The Floquet spectrum is a set of all k for which the transfer matrix $T(k)$ has the eigenvalues on the unit circle.

The next step is a determination of a Bloch solution χ of Eq. (3.1) as $\chi = MR$ which obeys the property $\chi(x + L, k) = \chi(x, k)\Delta(L, k)$, specific for the Bloch-type solutions. The Bloch eigenfunctions of the periodic spectral problem (3.1) play the role of the Jost solutions of the spectral problem with a decreasing potential.

Among the points of the Floquet spectrum we will distinguish the so called double points [22]. Double points are those values of k for which the Floquet exponents m_j differ in multipliers of $2\pi/L$ or, in other words, the Floquet multipliers are degenerate. We will be especially interested in complex double points which indicate linearized instability of solutions of Eq. (2.3) and label orbits homoclinic to hyperbolic fixed points in the phase space of a nonlinear system. Note that the term “double” in the context of the Floquet spectrum refers to the algebraic multiplicity of a point of the spectrum and has no relation to the multiplicity of zeros we have spoken about in the Introduction. Real double points are associated with stable modes.

Suppose we know explicitly a Bloch solution χ_0 of the spectral problem $\chi_{0x} = U_0\chi_0$ with the matrix U_0 (2.5) whose entries contain the known solution Q_0 of Eq. (2.4). Then we dress the solution χ_0 by applying the dressing factor $D(x, t, k)$, $\chi = D\chi_0$, and χ is a new solution of the spectral problem with new matrix $U = DU_0D^{-1} + D_xD^{-1}$. The dressing factor has the form

$$D = I - \sum_{s=1}^N \frac{k_s - k_s^*}{k - k_s^*} P_s(x, t), \quad (3.2)$$

where P_s is a projector, $P_s^2 = P_s$,

$$P_s = \frac{1}{k_s - k_s^*} \sum_{n,l=1}^{r_s} |n; s\rangle (D^{(s)-1})_{nl} \langle l; s|, \quad (3.3)$$

$$D_{nl}^{(s)} = \frac{\langle n; s | l; s \rangle}{k_s - k_s^*}.$$

Here k_s , $s = 1, \dots, N$ are complex double points of the Floquet spectrum and r_s is the rank of the projector P_s . The four-component ket- and bra-vectors $|n; s\rangle$ and $\langle l; s|$ are the column and row arrays, respectively. Hence, $|n; s\rangle$ is a four vector related with the sth complex double point k_s and obtained by applying the Bloch solution $\chi_0(k_s)$ to a constant vector $|q; s\rangle$,

$$|n; s\rangle = \chi_0(x, t, k_s) |q; s\rangle. \quad (3.4)$$

Exactly $r^{(s)}$ vectors $|q; s\rangle$, and hence $r^{(s)}$ vectors $|n; s\rangle$, correspond to the complex double point k_s . The summation in Eq. (3.2) is taken over all N complex double points, while that in Eq. (3.3) is performed over the r_s -dimensional space of vectors $|n; s\rangle$ produced in accordance with Eq. (3.4). Then a new solution of the matrix NLS equation is written as

$$\hat{Q} = \hat{Q}_0 + \sum_{s=1}^N (k_s - k_s^*) [\Lambda, P_s]. \quad (3.5)$$

For the rank 1 projectors these formulas reduce to the known ones [26].

Note the essential difference in applications of the dressing procedure between the soliton theory and the periodic wave theory. Indeed, the parameters k_s are free in the standard use of the dressing method and, in any case, they do not relate with the seed solution \hat{Q}_0 . On the contrary, our approach demands to choose k_s as the complex double points of the Floquet spectrum of the spectral problem (3.1) for the seed solution χ_0 .

Hence, they are the complex double points k_s and the projectors P_s that completely determine new solution. In the next section the above method will be used to generate homoclinic solution of the spin 1 BEC model (2.3) and hence to reveal the long-time dynamics of the MI in this model.

IV. RESULTS

We begin with a spatially homogeneous continuous wave solution of Eq. (2.3) with components

$$\phi_+^{(0)} = \phi_-^{(0)} = ae^{-i\mu t}, \quad \phi_0^{(0)} = ibe^{-i\mu t} \quad (4.1)$$

as the seed solution to be dressed. Here a and b are real constant amplitudes which determine a population of each spin component, and the chemical potential μ is given by $\mu = -2(a^2 + b^2)$. Note the fixed $\pi/2$ phase difference between the components $\phi_{\pm}^{(0)}$ and $\phi_0^{(0)}$. The same phase locking property is an inherent feature of the nonintegrable model (2.1) as well [10]. We could start with more general representation of plane waves but the structure of Eqs. (2.3) and the Galilean invariance reduce it to the form (4.1). Then we consider the spectral problem (3.1) with the matrix U_0 containing the plane waves (4.1) as the potential Q_0 :

$$U_0 = \begin{pmatrix} -ik & 0 & ae^{-i\mu t} & ibe^{-i\mu t} \\ 0 & -ik & ibe^{-i\mu t} & ae^{-i\mu t} \\ -ae^{i\mu t} & ibe^{i\mu t} & ik & 0 \\ ibe^{i\mu t} & -ae^{i\mu t} & 0 & ik \end{pmatrix}. \quad (4.2)$$

The fundamental solution of the spectral problem with the matrix U_0 is explicitly found:

$$M = \begin{pmatrix} \cos px + i(k/p) \sin px & 0 & (a/p) \sin pxe^{-i\mu t} & i(b/p) \sin pxe^{-i\mu t} \\ 0 & \cos px + i(k/p) \sin px & i(b/p) \sin pxe^{-i\mu t} & (a/p) \sin pxe^{-i\mu t} \\ -(a/p) \sin pxe^{i\mu t} & i(b/p) \sin pxe^{i\mu t} & \cos px - i(k/p) \sin px & 0 \\ i(b/p) \sin pxe^{i\mu t} & -(a/p) \sin pxe^{i\mu t} & 0 & \cos px - i(k/p) \sin px \end{pmatrix}, \quad \det M = 1, \quad (4.3)$$

where $p^2 = a^2 + b^2 + k^2$. Diagonalization of the transfer matrix $T(k) = M(L, k)$ is performed by the matrix R

which has the form

$$R = \begin{pmatrix} d_1 & -i(a/b)d_2 & [b/(p+k)]d_3e^{-i\mu t} & -i[a/(p+k)]d_4e^{-i\mu t} \\ -i(a/b)d_1 & d_2 & -i[a/(p+k)]d_3e^{-i\mu t} & [b/(p+k)]d_4e^{-i\mu t} \\ 0 & -(p-k)/b]d_2e^{i\mu t} & 0 & d_4 \\ -[(p-k)/b]d_1e^{i\mu t} & 0 & d_3 & 0 \end{pmatrix}, \quad (4.4)$$

where d_j are time dependent. As a result,

$$R^{-1}T(k)R = \Delta(L, k) = \text{diag}(e^{-ipL}, e^{-ipL}, e^{ipL}, e^{ipL}).$$

Therefore, the Floquet exponents are written as

$$m_1 = -p, \quad m_2 = -p, \quad m_3 = p, \quad m_4 = p \quad (4.5)$$

and have the multiplicity 2. Then we obtain the seed Bloch solution $\chi_0 = MR$ in the form

$$\chi_0 = \exp(i\frac{\mu}{2}t) \begin{pmatrix} d_{10} & -i\frac{a}{b}d_{20} & \frac{b}{p+k}d_{30} & -\frac{ia}{p+k}d_{40} \\ -i\frac{a}{b}d_{10} & d_{20} & -\frac{ia}{p+k}d_{30} & \frac{b}{p+k}d_{40} \\ 0 & \frac{k-p}{b}d_{20} & 0 & d_{40} \\ \frac{k-p}{b}d_{10} & 0 & d_{30} & 0 \end{pmatrix} \times \exp(ip\Lambda x + 2ikp\Lambda t), \quad (4.6)$$

where the parameters $d_j(t)$ entering Eq. (4.4) have been determined from the second Lax equation $\chi_{0t} = V_0\chi_0$ with the matrix V_0 (2.6) depending on the seed continuous wave (4.1):

$$d_1 = d_{10} \exp(-2ik^2t), \quad d_2 = d_{20} \exp(2ik^2t),$$

$$d_3 = d_{30} \exp\left[-\frac{i}{2}\mu t - 2ikpt\right], \quad d_4 = d_{40} \exp\left[\frac{i}{2}\mu t + 2ikpt\right].$$

Here d_{j0} are integration constants.

Now we proceed to finding the complex double points. Following Ref. [22], we seek for double points as a difference between two Floquet exponents m_1 and m_3 (4.5):

$$m_3 = m_1 + \delta_s, \quad \delta_s = \frac{2\pi}{L}s, \quad s = \pm 1, \pm 2, \dots$$

This gives

$$k_s = \pm i\sqrt{a^2 + b^2 - (\pi s/L)^2}, \quad \text{if } a^2 + b^2 > (\pi s/L)^2 \quad (4.7)$$

and

$$k_s = \pm \sqrt{(\pi s/L)^2 - (a^2 + b^2)}, \quad \text{if } a^2 + b^2 < (\pi s/L)^2. \quad (4.8)$$

Hence, for given amplitudes a and b and period L the double points are arranged into infinite number of real double points (4.8) situated on the real axis in the k plane, and a finite number of complex double points (4.7) lying on the imaginary axis within the interval $(-i\sqrt{a^2 + b^2}, i\sqrt{a^2 + b^2})$.

Let us choose in the following the amplitudes and period in such a way that to obtain the single complex double point k_1 (and hence $-k_1$). It means $\sqrt{a^2 + b^2} > (\pi/L)$ but $\sqrt{a^2 + b^2} < (2\pi/L)$. For this choice the only rank 2 projector $P \equiv P_1$ has the form

$$P = \frac{1}{k_1 - k_1^*} \sum_{n,l=1}^2 |n\rangle (D^{-1})_{nl} \langle l|, \quad D_{nl} = \frac{\langle n|l\rangle}{k_1 - k_1^*}.$$

To simplify notations, we write $|n\rangle$ instead of $|n; 1\rangle$. Since D is a 2×2 matrix, we easily obtain the following expression for the projector:

$$P = \frac{1}{\tilde{D}} [\langle 2|2\rangle |1\rangle \langle 1| - \langle 1|2\rangle |1\rangle \langle 2| - \langle 2|1\rangle |2\rangle \langle 1| + \langle 1|1\rangle |2\rangle \langle 2|], \quad (4.9)$$

$$\tilde{D} = \langle 1|1\rangle \langle 2|2\rangle - \langle 1|2\rangle \langle 2|1\rangle,$$

where the vectors $|1\rangle$ and $|2\rangle$ are determined as

$$|1\rangle = \chi_0(k_1)|q\rangle, \quad |2\rangle = \chi_0(k_1)|r\rangle.$$

Here $|q\rangle$ and $|r\rangle$ are linearly independent constant vectors with the components q_j and r_j , $j = 1, \dots, 4$. Then after rather lengthy but straightforward algebraic calculation in accordance with Eqs. (4.9) and (3.5) taken for $s = 1$, we explicitly obtain new solutions of the integrable spin 1 BEC model (2.3),

$$\phi_+ = \phi_- = ae^{-i\mu t} \left(1 + 2i \frac{B}{A} \sin \psi \right), \quad (4.10)$$

$$\phi_0 = ibe^{-i\mu t} \left(1 + 2i \frac{B_0}{A} \sin \psi \right),$$

which at the same time represent components of the matrix homoclinic orbit of the matrix NLS equation (2.4). Here

$$A = \cosh 2\tau - \cos 2\rho \sin^2 \psi + 2\gamma \cosh \tau \sin \rho \sin \psi + \frac{1}{2}\gamma^2, \quad (4.11)$$

$$\begin{aligned} B &= \sinh 2\tau \cos \psi + i \cosh 2\tau \sin \psi - i \cos 2\rho \sin \psi \\ &+ \frac{1}{4}\gamma [(\mu_- e^\tau + \mu_-^* e^{-\tau})e^{i\rho} - (\mu_+^* e^\tau + \mu_+ e^{-\tau})e^{-i\rho}] \\ &+ \frac{i}{2}\gamma^2 \left(\sin \psi - \frac{b}{a} \cos \psi \cos \alpha_{23} \right), \end{aligned} \quad (4.12)$$

$$\begin{aligned} B_0 &= \sinh 2\tau \cos \psi + i \cosh 2\tau \sin \psi - i \cos 2\rho \sin \psi \\ &+ \frac{1}{4}\gamma [(\nu_- e^\tau + \nu_-^* e^{-\tau})e^{i\rho} - (\nu_+^* e^\tau + \nu_+ e^{-\tau})e^{-i\rho}] \\ &+ \frac{i}{2}\gamma^2 \left(\sin \psi - \frac{a}{b} \cos \psi \cos \alpha_{23} \right). \end{aligned} \quad (4.13)$$

The constants d_{j0} have been incorporated into the constant components q_j and r_j of the vectors $|q\rangle$ and $|r\rangle$. If we denote definite combinations of these components as

$$e_1 = q_1 r_2 - q_2 r_1, \quad e_2 = q_1 r_3 - q_3 r_1, \quad e_3 = q_1 r_4 - q_4 r_1,$$

$$e_4 = q_3 r_4 - q_4 r_3, \quad e_j = |e_j| e^{i\alpha_j}, \quad \alpha_{jl} = \alpha_j - \alpha_l, \quad |e_2| = |e_3|,$$

then the notations used in (4.11), (4.12) and (4.13) are as follows:

$$k_1 = ik_0, \quad p_1 = \sqrt{a^2 + b^2 + k_1^2} = \frac{\pi}{L}, \quad e^{i\psi} = \frac{p_1 + ik_0}{\sqrt{a^2 + b^2}},$$

$$\tau = 4k_0 p_1 t + t_0, \quad \rho = 2p_1 x - \alpha_{13}, \quad \alpha_{13} = \alpha_{34},$$

$$e^{t_0} = \frac{\sqrt{a^2 + b^2}}{b} \sqrt{\frac{|e_1|}{|e_4|}}, \quad \gamma = \frac{2|e_2|}{\sqrt{|e_1||e_4|}},$$

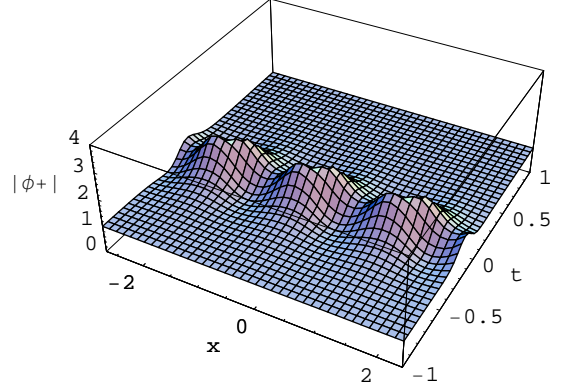


FIG. 1: Full-time evolution of the ϕ_+ (and ϕ_-) component due to modulational instability. The parameters are $a = 1$, $b = 2$, $L = \pi/2$, $\alpha_2 = \pi/3$, $\alpha_3 = \pi/4$, $|e_j| = 1$.

$$\mu_{\pm} = 1 \pm ie^{\mp i\psi} \left(\sin \psi - \frac{b}{a} e^{i\alpha_{23}} \cos \psi \right),$$

$$\nu_{\pm} = 1 \pm ie^{\mp i\psi} \left(\sin \psi + \frac{a}{b} e^{i\alpha_{23}} \cos \psi \right).$$

The solutions (4.10) are indeed homoclinic to the plane waves (4.1). Calculation of the asymptotics of ϕ_{\pm} and ϕ_0 as $t \rightarrow \pm\infty$ gives

$$\phi_{\pm} \rightarrow ae^{-i\mu t} e^{\pm 2i\psi}, \quad \phi_0 \rightarrow ibe^{-i\mu t} e^{\pm 2i\psi}.$$

In other words, these solutions reproduce in the limit $t \rightarrow \pm\infty$ the seed plane waves up to a constant phase, as should be for the homoclinic orbit. Note that the nonlinear MI for the spin 1 condensate but for different phases of the seed wave components was studied by the Darboux transformation in Ref. [14].

Figures 1 and 2 illustrating the solution (4.10) demonstrate typical development of the continuous wave perturbation within three periods in x . We see that the stage of the exponential growth of instabilities revealed by the linear stability analysis transforms to the exponential decreasing with emergence of localized structures. Hence, the full-time evolution of MI for the integrable $F = 1$ BEC model demonstrates the reversal property, such as the Fermi–Pasta–Ulam process [30]: the phase trajectory of the system returns to the initial one which corresponds to the continuous waves (4.1). For chosen parameters the growth and decrease development of the component ϕ_0 is more pronounced than that of ϕ_{\pm} .

V. CONCLUSIONS

We have derived the analytic formulas for describing the full-time dynamics of the modulational instability in

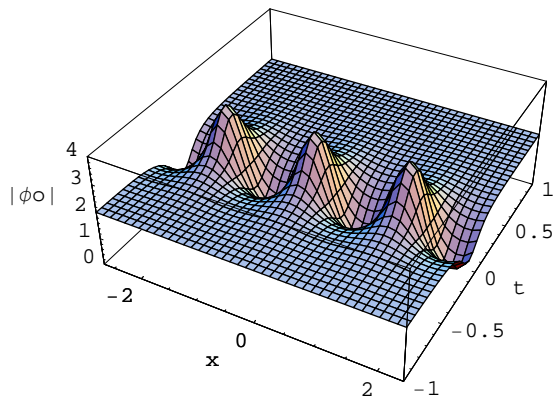


FIG. 2: Full-time evolution of the ϕ_0 component due to modulational instability. The parameters are the same as in Fig. 1.

the integrable model of $F = 1$ Bose-Einstein condensates. Our results are based on the exact homoclinic solution of the matrix NLS equation with the continuous plane waves as an initial condition. We have shown that there exist cycles of the MI evolution with the reversal property when the exponential growth of the modulation amplitude changes to its exponential decay. The solution we present here is an example of *large-amplitude* periodic solutions. It describes an exponential growth of

a weak modulation of a background for an initial stage of the condensate evolution, and in this sense the background is unstable. However, in the nonlinear regime this exponentially growing mode saturates and subsequently transforms into oscillations. As expected, the integrable model (2.3) does not exhibit long-time chaotic dynamics contrary to the regimes observed numerically for a general case [10], but it may serve as a good analytical approximation of the evolution of the condensate experiencing the instability. Higher-order homoclinic solutions which correspond to several complex double points can be obtained by the method described in [26].

Strictly speaking, the analysis based on the continuous wave model is not applicable to the trapped systems. Nevertheless, such an approach remains valid when the typical spatial extent of the condensate is larger than the period of the localized pattern formed in result of the instability. More realistic models should account for a (small) deviation of the condensate parameters from the constraint which provides integrability of the model. There exists an approach [31] to reveal a persistence of the homoclinic orbit when the integrability condition breaks, and therefore to establish analytically the existence of chaos. This approach is based on the construction of the so-called Melnikov function from the quadratic products of the Bloch functions evaluated on the homoclinic orbit. In this paper we have explicitly built all the ingredients to perform the Melnikov analysis. Corresponding results will be published elsewhere.

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